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Journal of Computational and Applied Mathematics 185 (2006) 1–18
JOURNAL OF COMPUTATIONAL AND APPLIED MATHEMATICS
www.elsevier.com/locate/cam

Long-time error estimation for semi-linear parabolic equations

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Received 22 April 2003; received in revised form 23 December 2004

Abstract

The long-time error estimation approach of Sun and Ewing (Dyn. Contin. Discrete Impuls. Systems Ser. B Appl. Algorithms, 9 (2002) 115–129) is applied here for the error analysis and estimation of linear and semi-linear parabolic partial differential equations. The analysis is carried out using the stability–smoothing indicator, the smoothing assumption, the moving attractor, the exact error propagation and the two-level error propagation analysis introduced by Sun and Ewing (Dyn. Contin. Discrete Impuls. Systems Ser. B Appl. Algorithms, 9 (2002) 115–129). Moreover, an inverse elliptic projection is employed here as a key technique in dealing with the spatial discretization error. The error estimates obtained are uniform in time. The results are substantiated by a complete mathematical analysis and numerical experiments.

MSC: 65M15; 65L20

Keywords: Long-time error estimation; Stability–smoothing indicator; Moving attractor; Semi-linear parabolic PDE; Numerical solution

1. Introduction

In this paper we propose a new approach to estimating the error of the numerical solutions of a family of semi-linear parabolic equations. This approach features the exact error propagation and the stability–smoothing indicator.

The initial motivation of this research is to avoid numerical error propagation in the error analysis. To do so, one has to, and only needs to, deal with the smoothing property of the semi-discrete and the fully

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doi:10.1016/j.cam.2005.01.033

discrete numerical schemes. This can be understood by studying the error splitting in the proof of our main theorem. What makes this approach work is that the smoothing property of the numerical scheme can be monitored by the stability–smoothing indicator, while the smoothing property of the semi-discrete scheme only needs to be proven locally within a time step.

In this approach, we only impose the essential conditions on numerical schemes: consistency, stability and smoothing property. Consistency is usually easy to study. Stability and smoothing property are obtained by using a stability–smoothing indicator. Other than that, the analysis does not rely on any details of a particular scheme. Therefore, one shall expect no technical difficulty with this approach in applying it to complicated numerical schemes involving local time stepping, domain decomposition, different algorithms on different parts of a multi-variable system, etc. The computing cost of the stability–smoothing indicator in each step is, at most, comparable to the cost of solving an implicit scheme in the step. One can actually decide the step size of the next step by the value of the indicator on the current node (t_i).

The error estimate in our main theorem depends heavily on the a priori knowledge of a few constants, namely, they are the contraction rate of a moving attractor, the one-sided Lipschitz constant, and the constants in the local error estimates. This leads to three possible situations: (1) Without knowing these constants, one can use the error estimate of the main theorem as a convergence result. (2) If one knows those constants, but does not have any a priori estimates on the stability–smoothing indicator, one can compute the stability–smoothing indicator and the theorem provides a posteriori error estimate. (3) If one knows those constants and can prove that the stability–smoothing indicator is bounded by a given number, then the theorem serves as a priori error estimate.

A list of papers is given in Refs. [2–7,10], representing some of the other research approaches and results in the area of numerical solutions of nonlinear parabolic differential equations and dynamical systems.

The outline of the paper is as follows. In Section 2, we present the semi-linear parabolic problems. We also introduce some notations and properties of the solutions, in preparation for the succeeding analysis. In Section 3, we collect some of the theoretical results on semi-linear parabolic equations that are needed in the error analysis. In Section 4, we proceed to present a family of semi-discrete and fully discrete finite element methods, followed by the stability and smoothing analyses for the semi-discrete problems. In Section 5, we introduce the concepts of a moving attractor and a stability–smoothing indicator. They will play major roles in the long-time error estimation. In Section 6, the main theorem on long-time error estimation is stated and proved. Finally, the results of a few preliminary numerical experiments are reported in Section 7.

2. Problem and preliminaries

Consider the semi-linear parabolic problem

$$\dot{\mathbf{u}} = \Delta \mathbf{u} + R(\mathbf{u}) \quad \text{in } \Omega \tag{2.1}$$

in a convex polygonal domain $\Omega \subset \mathbf{R}^2$, with the Dirichlet boundary condition

$$\mathbf{u} = 0 \quad \text{on } \partial\Omega$$

for $t \in [t_0, \infty)$ and the initial condition

$$\mathbf{u}(t_0) = \mathbf{u}_0.$$

The homogeneous boundary value is just for simplicity. More general boundary values can be treated in the framework of the paper. In all the notations for the solutions of (2.1) in this paper, the spatial variable $x \in \Omega$ is not explicitly written. Function $R : \mathbf{R} \to \mathbf{R}$ in (2.1) is at least twice differentiable.

For convenience of error propagation analysis in the following sections, we use the notation of a dynamical system for the solutions of Eq. (2.1). That is,

$$\mathbf{u}(p, t, \mathbf{v}) \tag{2.2}$$

stands for the value of the solution of Eq. (2.1) at time $t + p$ with initial time t, initial value $\mathbf{v} \in L^2(\Omega)$ and time increment p. With this notation, the well-known semi-group property can be written as

$$\mathbf{u}(p + r, t, \mathbf{v}) = \mathbf{u}(p, t + r, \mathbf{u}(r, t, \mathbf{v})).$$

Here we could have used a semi-group action operator $S(p)$, with $\mathbf{u}(p, t, \mathbf{v}) = S(p)\mathbf{v}$. The notation of (2.2) is consistent with that in [11]. The reason for choosing this notation is that it also applies to nonautonomous equations.

3. Existence, uniqueness and stability

In order to solve the problem numerically and to estimate the error, we certainly need the qualititive results on differential equation (2.1). Under proper conditions, one can prove that there exists a unique solution for Eq. (2.1) with any given initial value in a proper space. In addition, the solution can be proven to exist in $[t_0, \infty)$, it is smooth, stable, and there is an exponential attractor. For the convenience of the reader, we include some of these results in this section.

In this paper, we will use the standard Banach spaces $L_p = L_p(\Omega)$ and the standard Sobolev spaces $H^m = H^m(\Omega)$. The norm for functions in $L_p(\Omega)$ is

$$\|u\|_{L_p} = \left(\int_\Omega |u|^p \, \mathrm{d}\Omega \right)^{1/p}.$$

For $p = 2$, we use the simplified notation $\|u\| = \|u\|_{L_2}$. The inner product on $L_2(\Omega)$ is denoted by

$$(u, v) = \int_\Omega uv \, \mathrm{d}\Omega.$$

The norm for functions in $H^m(\Omega)$ is

$$\|u\|_m = \left(\sum_{|\alpha| \leqslant m} \|D^\alpha u\|^2 \right)^{1/2}.$$

We will also use the standard Sobolev space with the homogeneous boundary condition,

$$H_0^1(\Omega) = \{u \in H^1(\Omega) : \forall x \in \partial\Omega, u(x) = 0\}.$$

3.1. Local existence

For local existence results, we refer to the lecture notes by Larsson [8]. While there are many other possible references, such as [9], Larsson's lecture notes deal with convex polygonal domains directly, which is convenient for finite element analysis.

Theorem 3.1. *Assume that $R : \mathbf{R} \to \mathbf{R}$ is at least twice continuously differentiable and*

$$|R^{(l)}(\xi)| \leqslant C(1 + |\xi|^{\delta+1-l}) \tag{3.1}$$

for a real number $\delta \in [1, \infty)$, $l = 0, 1$ or 2, and any $\xi \in \mathbf{R}$. For any given $R_0 > 0$, let $\mathscr{B} = \{\mathbf{u} \in H_0^1(\Omega) : \|\mathbf{u}\|_1 \leqslant R_0\}$. There is a positive real number $t_L = t_L(R_0)$, such that, for any initial value $\mathbf{u}_0 \in \mathscr{B}$, there exists a unique solution $\mathbf{u}(t)$ of (2.1) *in $[0, t_L]$, with $\|\mathbf{u}(t)\|_1 \leqslant C R_0$ for all $t \in [0, t_L]$.*

Proof. See [8, Theorem 1.2], for the proof. □

3.2. Global existence

For the global existence of a solution, we consider the concept of a invariant region [9]. With this, the possibility of any finite time blow-up has been excluded.

Theorem 3.2. *If there are R_- and R_+, with $R_- < R_+$, such that $R(R_-) > 0$ and $R(R_+) < 0$, then for any initial value $\mathbf{u}_0 \in \mathscr{B}$ with $R_- \leqslant \mathbf{u}_0 \leqslant R_+$, the solution $\mathbf{u}(t)$ exists in $[t_0, \infty)$, with $R_- \leqslant \mathbf{u}(t) \leqslant R_+$ for all t.*

Proof. See [9, Chapter 14, Section B] for a more general case and its proof. □

3.3. Local stability

For the local stability of a nonlinear problem, we usually cannot expect having a monotone operator. Instead, we assume the one-sided Lipschitz condition.

Theorem 3.3. *Let $\tilde{\mathscr{B}}$ be a bounded subset of $H_0^1(\Omega)$. In addition, assume that $\tilde{\mathscr{B}}$ is positively invariant and $R_- \leqslant \mathbf{u}(t) \leqslant R_+$ for all $\mathbf{u}(t)$* in *$\tilde{\mathscr{B}}$ and all $t > t_0$. If the one-sided Lipschitz condition*

$$(\Delta\mathbf{u} + R(\mathbf{u}) - \Delta\mathbf{v} - R(\mathbf{v}), \mathbf{u} - \mathbf{v}) \leqslant m\|\mathbf{u} - \mathbf{v}\|^2 \tag{3.2}$$

is satisfied for all $\mathbf{u}$ and $\mathbf{v}$ in *$\tilde{\mathscr{B}}$, then, for any initial values $\mathbf{u}(t_0) = \mathbf{u}_0$ and $\mathbf{v}(t_0) = \mathbf{v}_0$ in $\tilde{\mathscr{B}}$, the corresponding solutions $\mathbf{u}(t)$ and $\mathbf{v}(t)$ satisfy*

$$\|\mathbf{u}(t) - \mathbf{v}(t)\| \leqslant \mathrm{e}^{m(t-t_0)}\|\mathbf{u}_0 - \mathbf{v}_0\|. \tag{3.3}$$

Proof. Since

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{u}-\mathbf{v}\|^2 &= \left(\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{u}-\mathbf{v}), \mathbf{u}-\mathbf{v}\right)\\ &= (\Delta\mathbf{u}+R(\mathbf{u})-\Delta\mathbf{v}-R(\mathbf{v}), \mathbf{u}-\mathbf{v})\\ &\leqslant m(\mathbf{u}-\mathbf{v}, \mathbf{u}-\mathbf{v})\\ &= m\|\mathbf{u}-\mathbf{v}\|^2,\end{aligned}$$

we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{u}-\mathbf{v}\| \leqslant m\|\mathbf{u}-\mathbf{v}\|,$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{e}^{-mt}\|\mathbf{u}-\mathbf{v}\|) \leqslant -m\mathrm{e}^{-mt}\|\mathbf{u}-\mathbf{v}\| + \mathrm{e}^{-mt}\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{u}-\mathbf{v}\| \leqslant 0.$$

Therefore,

$$\mathrm{e}^{-mt}\|\mathbf{u}(t)-\mathbf{v}(t)\| \leqslant \mathrm{e}^{-mt_0}\|\mathbf{u}(t_0)-\mathbf{v}(t_0)\|$$

and (3.3) is proven. □

3.4. Exponential attractor

Theorem 3.4. *Under the conditions in the previous theorems, Eq.* (2.1) *admits an exponential attractor* $\mathscr{M} \subset L^2(\Omega)$. *That is, there exist* $s > 0$, $\theta \in (0, 1)$, *and a positively invariant subset* $\mathscr{U}$ *of* $L^2(\Omega)$, *such that, for any initial value* $\mathbf{u}_0 \in \mathscr{U}$,

$$d(\mathbf{u}(t+s), \mathscr{M}) \leqslant \theta d(\mathbf{u}(t), \mathscr{M})$$

for all $t > t_0$. *In addition, the fractal dimension of* $\mathscr{M}$ *is finite.* $\mathscr{U}$ *is called a basin.*

Proof. See [1, Chapter 3] for the proof of this theorem. □

Remark 3.5. The terminology “fractal dimension” and “positively invariant” can be found in [1]. We will not use the finiteness of fractal dimension in this paper. Positively invariant one-parameter family of sets under a dynamical system will be defined in Section 5.

In [1], the parameters s and θ are given in the form

$$\theta = a_0\mathrm{e}^{-a_1 s}$$

with a_0 and a_1 positive. For a sufficiently large s, θ is less than 1. Sharp estimation of a_0 and a_1 is a challenging research topic. But any available estimation of a_0 and a_1 can be used for the error control of a numerical solution.

3.5. Smoothing action

Theorem 3.6. *Let $R_1 > 0$ and $t_L > 0$ be given and $\mathbf{u} \in C([0, t_L], H_0^1(\Omega))$ be a solution of* (2.1). *If $\|\mathbf{u}(t)\|_1 \leqslant R_1$ for all $t \in [0, t_L]$, then*

$$\|\mathbf{u}(t)\|_2 \leqslant C(R_1, t_L) t^{-1/2}, \tag{3.4}$$

$$\|\mathbf{u}_t(t)\|_s \leqslant C(R_1, t_L) t^{-1-(s-1)/2}, \quad s = 0, 1, 2. \tag{3.5}$$

Proof. See [8, Theorem 1.3], for the proof. □

4. The finite element methods

Instead of the differential equation (2.1), we will consider its weak formulation: Find $\mathbf{u}(t) \in C^1([t_0, \infty), H_0^1(\Omega))$, such that

$$\left(\frac{\partial \mathbf{u}}{\partial t}, \mathbf{v}\right) + (\nabla \mathbf{u}, \nabla \mathbf{v}) = (R(\mathbf{u}), \mathbf{v}) \tag{4.1}$$

for all $\mathbf{v} \in H_0^1(\Omega)$.

Let $\mathscr{T}_h$ be a quasi-uniform triangulations of Ω, where h is the characteristic mesh size of $\mathscr{T}_h$,

$$h = \max\{\mathrm{diam}(T_i) | T_i \in \mathscr{T}_h\}.$$

Let $V_{h,p}$ be the finite element space consisting of continuous piecewise polynomials of order p:

$$V_{h,p} = \{q \in H_0^1(\Omega) : q(x)|_{x \in T_i} \in \mathscr{P}_p(T_i)\},$$

where $\mathscr{P}_p(T_i)$ is the set of all the polynomials in T_i up to order p. When it is clear that the order is p in the context, we use V_h for $V_{h,p}$. In this paper, the error estimation is only for the case $p = 1$. The auxiliary function defined in (6.2) only works for $p = 1$ currently. It will take further research to generalize the result.

The semi-discrete approximation $\mathbf{u}_h \in C^1([t_0, \infty), V_h)$ of a solution $\mathbf{u}$ of (4.1) is determined by

$$\left(\frac{\partial \mathbf{u}_h}{\partial t}, \mathbf{v}\right) + (\nabla \mathbf{u}_h, \nabla \mathbf{v}) = (R(\mathbf{u}_h), \mathbf{v}) \tag{4.2}$$

for all $\mathbf{v} \in V_h$. The initial value of (4.2) the elliptic projection of the initial value of (4.1).

For simplicity, we use a fixed time step size τ for the discretization of time. A single step finite difference method will be used, such as the backward Euler method, a prediction–correction method, an explicit or implicit Runge–Kutta method, a discrete Galerkin method, or any more complicated schemes. Within each step, one can use local time stepping or a combination of different schemes on different parts of the ODE resulting from (4.2). The only two requirements are that (1) the local error can be proven to be of order τ^2 or τ^3 under the condition that the second or third time derivative is bounded in L^2 norm, and (2) the scheme is stable and smoothing in the sense of Definition 5.1. Instead of one specified numerical scheme, here we admit a family of finite element spaces and a family of discretization methods of time.

For the numerical solution of the fully discrete scheme, we use the notation $\mathbf{u}_N(t)$. In error propagation analysis, we will need to write the numerical solution as $\mathbf{u}_N(p, t, \mathbf{v})$, mimicking those notations we introduced in Section 2 for the solutions of the PDE (2.1) or the solutions of the weak formulation (4.1). In $\mathbf{u}_N(p, t, \mathbf{v})$, t is the initial time, $\mathbf{v}$ is the initial value at time t, and p is the time increment. One can easily observe that the semi-group property

$$\mathbf{u}_N(p + r, t, \mathbf{v}) = \mathbf{u}_N(p, t + r, \mathbf{u}_N(r, t, \mathbf{v}))$$

is valid as long as p, r and $t - t_0$ are multiples of the time step τ, and a single-step method is used.

Similarly, we use the notation $\mathbf{u}_h(p, t, \mathbf{v})$ for the semi-discrete solution of (4.2) with initial time t, initial value $\mathbf{v}$ and time increment p. For the semi-discrete solutions, it is easy to verify that the semi-group property remains valid,

$$\mathbf{u}_h(p + r, t, \mathbf{v}) = \mathbf{u}_h(p, t + r, \mathbf{u}_h(r, t, \mathbf{v})).$$

For the short time stability of the semi-discrete solutions, we state the following theorem.

Theorem 4.1. *Let $\tilde{\mathscr{B}}$ be defined as in the last section. For any $\mathbf{u}_0$ and $\mathbf{v}_0$ in $\tilde{\mathscr{B}}$, the corresponding semi-discrete solutions $\mathbf{u}_h(t)$ and $\mathbf{v}_h(t)$ satisfy*

$$\|\mathbf{u}_h(t) - \mathbf{v}_h(t)\| \leqslant \mathrm{e}^{m(t-t_0)} \|\mathbf{u}_0 - \mathbf{v}_0\|. \tag{4.3}$$

Proof. It is easy to verify that the one-sided Lipschitz condition (3.2) is also satisfied by the functions in the finite element space. Moreover, the proof of Theorem 3.3 can be used word by word in the semi-discrete case. □

The next theorem is crucial for the estimation of the local error resulting from the discretization of time. For the proof of this theorem and the definitions in the next section, we need to introduce the discrete Laplace operator $\Delta_h : H_0^1(\Omega) \longrightarrow V_h$ defined by

$$(\Delta_h \mathbf{u}, \mathbf{v}) = -(\nabla \mathbf{u}, \nabla \mathbf{v}) \quad \forall \mathbf{v} \in V_h,$$

and the L^2 projection operator $P_h : L^2(\Omega) \longrightarrow V_h$ by

$$(P_h \mathbf{u}, \mathbf{v}) = (\mathbf{u}, \mathbf{v}) \quad \forall \mathbf{v} \in V_h.$$

Theorem 4.2. *For any initial value $\bar{\mathbf{u}} \in V_h$, if*

$$\bar{\mathbf{v}} = \Delta_h \bar{\mathbf{u}} + P_h R(\bar{\mathbf{u}}),$$

$$\bar{\mathbf{w}} = \Delta_h \bar{\mathbf{v}} + P_h (R'(\bar{\mathbf{u}})\bar{\mathbf{v}}),$$

and there is a constant $\bar{C}$ such that $\|\bar{\mathbf{u}}\|_1 \leqslant \bar{C}$, $\|\bar{\mathbf{v}}\| \leqslant \bar{C}$, $\|\bar{\mathbf{w}}\| \leqslant \bar{C}$, then the corresponding semi-discrete solution $\mathbf{u}_h(p, t, \bar{\mathbf{u}})$ satisfies

$$\left\| \frac{\partial^2}{\partial p^2} \mathbf{u}_h(p, t, \mathbf{v}) \right\| \leqslant C_M + C_0 \|\bar{\mathbf{u}}\| + C_0 \|\bar{\mathbf{v}}\| + C_2 \|\bar{\mathbf{w}}\| \tag{4.4}$$

for sufficiently small p and some constants C_M, C_0, C_1 and C_2.

Remark. The reason for the boundedness of $\|\bar{\mathbf{u}}\|_1$, $\|\bar{\mathbf{v}}\|$ and $\|\bar{\mathbf{w}}\|$ can be found in the next section. In this paper, a time step size τ of a fully discrete scheme will be required to be less than the "sufficiently small" p given here.

Proof. Let $\mathbf{v}_h = \mathrm{d}\mathbf{u}_h/\mathrm{d}p$ and $\mathbf{w}_h = \mathrm{d}^2\mathbf{u}_h/\mathrm{d}p^2$. From (4.2), it is easy to verify that $\mathbf{u}_h$, $\mathbf{v}_h$ and $\mathbf{w}_h$ satisfy

$$\left(\frac{\partial \mathbf{u}_h}{\partial p}, \mathbf{z}_0\right) + (\nabla \mathbf{u}_h, \nabla \mathbf{z}_0) = (R(\mathbf{u}_h), \mathbf{z}_0), \tag{4.5}$$

$$\left(\frac{\partial \mathbf{v}_h}{\partial p}, \mathbf{z}_1\right) + (\nabla \mathbf{v}_h, \nabla \mathbf{z}_1) = (R'(\mathbf{u}_h)\mathbf{v}_h, \mathbf{z}_1), \tag{4.6}$$

$$\left(\frac{\partial \mathbf{w}_h}{\partial p}, \mathbf{z}_2\right) + (\nabla \mathbf{w}_h, \nabla \mathbf{z}_2) = (R'(\mathbf{u}_h)\mathbf{w}_h + R''(\mathbf{u}_h)\mathbf{v}_h^2, \mathbf{z}_2) \tag{4.7}$$

for all $\mathbf{z}_0$, $\mathbf{z}_1$ and $\mathbf{z}_2 \in V_h$. It is easy to observe that the initial values are $\mathbf{u}_h(t)=\bar{\mathbf{u}}$, $\mathbf{v}_h(t)=\bar{\mathbf{v}}$, and $\mathbf{w}_h(t)=\bar{\mathbf{w}}$.

Using [8, Theorem 2.1] on (4.5), we can prove that, for sufficiently small p,

$$\|\mathbf{u}_h(t+p)\|_1 \leqslant C(1 + \|\bar{\mathbf{u}}\|_1) \tag{4.8}$$

for a constant C. Similarly, using [8, proof of Theorem 2.1] on (4.6), we can prove that, for sufficiently small p,

$$\|\mathbf{v}_h(t+p)\|_1 \leqslant C(1 + \|\bar{\mathbf{v}}\|_1). \tag{4.9}$$

Replacing $\mathbf{z}_0$ by $\mathbf{u}_h$ in (4.5), $\mathbf{z}_1$ by $\mathbf{v}_h$ in (4.6) and $\mathbf{z}_2$ by $\mathbf{w}_h$ in (4.7), we obtain

$$\left(\frac{\partial \mathbf{u}_h}{\partial p}, \mathbf{u}_h\right) + (\nabla \mathbf{u}_h, \nabla \mathbf{u}_h) = (R(\mathbf{u}_h), \mathbf{u}_h),$$

$$\left(\frac{\partial \mathbf{v}_h}{\partial p}, \mathbf{v}_h\right) + (\nabla \mathbf{v}_h, \nabla \mathbf{v}_h) = (R'(\mathbf{u}_h)\mathbf{v}_h, \mathbf{v}_h),$$

$$\left(\frac{\partial \mathbf{w}_h}{\partial p}, \mathbf{w}_h\right) + (\nabla \mathbf{w}_h, \nabla \mathbf{w}_h) = (R'(\mathbf{u}_h)\mathbf{w}_h + R''(\mathbf{u}_h)\mathbf{v}_h^2, \mathbf{w}_h).$$

Adding these three together

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}p}(\|\mathbf{u}_h\|^2 + \|\mathbf{v}_h\|^2 + \|\mathbf{w}_h\|^2) + (\|\mathbf{u}_h\|_1^2 + \|\mathbf{v}_h\|_1^2 + \|\mathbf{w}_h\|_1^2) \\
&\quad = (R(\mathbf{u}_h), \mathbf{u}_h) + (R'(\mathbf{u}_h)\mathbf{v}_h, \mathbf{v}_h) + (R'(\mathbf{u}_h)\mathbf{w}_h + R''(\mathbf{u}_h)\mathbf{v}_h^2, \mathbf{w}_h) \\
&\quad \leqslant \|R(\mathbf{u}_h)\|\|\mathbf{u}_h\| + \|R'(\mathbf{u}_h)\mathbf{v}_h\|\|\mathbf{v}_h\| + \|R'(\mathbf{u}_h)\mathbf{w}_h + R''(\mathbf{u}_h)\mathbf{v}_h^2\|\|\mathbf{w}_h\|.
\end{aligned} \tag{4.10}$$

Since the domain Ω is in $\mathbf{R}^2$ and $R(u)$ satisfies (3.1), by using the Sobolev inequality, we have

$$\begin{aligned}
\|R(\mathbf{u}_h)\|^2 &= \int_\Omega R^2(\mathbf{u}_h)\,\mathrm{d}\Omega \leqslant C\int_\Omega (1 + |\mathbf{u}_h|^{1+\delta})^2\,\mathrm{d}\Omega \\
&\leqslant C(1 + \|\mathbf{u}_h\|_{L_{2+2\delta}}^{2+2\delta}) \leqslant C(1 + \|\mathbf{u}_h\|_1^{2+2\delta})
\end{aligned}$$

and hence, with the help of (4.8),

$$\|R(\mathbf{u}_h(t+p))\| \leqslant C(1+\|\mathbf{u}_h(t+p)\|_1^{1+\delta}) \leqslant C(1+\|\bar{\mathbf{u}}\|_1^{1+\delta}). \tag{4.11}$$

Similarly, one can show that

$$\begin{aligned}\|R'(\mathbf{u}_h)\mathbf{v}_h\|^2 &= \int_\Omega R'(\mathbf{u}_h)^2\mathbf{v}_h^2\,\mathrm{d}\Omega \leqslant C\int_\Omega (1+|\mathbf{u}_h|^\delta)^2\mathbf{v}_h^2\,\mathrm{d}\Omega\\ &\leqslant C(1+\|\mathbf{u}_h\|_{L_{2\delta}}^{2\delta})\|\mathbf{v}_h\|_{L_4}^2 \leqslant C(1+\|\mathbf{u}_h\|_1^{2\delta})\|\mathbf{v}_h\|_1^2,\end{aligned}$$

and hence,

$$\|R'(\mathbf{u}_h(t+p))\mathbf{v}_h(t+p)\| \leqslant C(1+\|\bar{\mathbf{u}}\|_1^\delta)\|\mathbf{v}_h(t+p)\|_1. \tag{4.12}$$

In exactly the same way, one can also prove

$$\|R'(\mathbf{u}_h(t+p))\mathbf{w}_h(t+p)\| \leqslant C(1+\|\bar{\mathbf{u}}\|_1^\delta)\|\mathbf{w}_h(t+p)\|_1. \tag{4.13}$$

Now, repeating the same technique, we get

$$\begin{aligned}\|R''(\mathbf{u}_h)\mathbf{v}_h^2\|^2 &= \int_\Omega R''(\mathbf{u}_h)^2\mathbf{v}_h^4\,\mathrm{d}\Omega \leqslant C\int_\Omega (1+|\mathbf{u}_h|^{\delta-1})^2\mathbf{v}_h^4\,\mathrm{d}\Omega\\ &\leqslant C(1+\|\mathbf{u}_h\|_{L_{2\delta-2}}^{2\delta-2})\|\mathbf{v}_h\|_{L_8}^4 \leqslant C(1+\|\mathbf{u}_h\|_1^{2\delta-2})\|\mathbf{v}_h\|_1^4,\end{aligned}$$

hence, by using (4.8) and (4.9),

$$\|R''(\mathbf{u}_h(t+p))\mathbf{v}_h^2(t+p)\| \leqslant C(1+\|\bar{\mathbf{u}}\|_1^{\delta-1})(1+\|\bar{\mathbf{v}}\|_1)\|\mathbf{v}_h(t+p)\|_1, \tag{4.14}$$

where $\|\bar{\mathbf{v}}\|_1$ can be bounded in terms of $\|\bar{\mathbf{u}}\|_1$, $\|\bar{\mathbf{v}}\|$, and $\|\bar{\mathbf{w}}\|$, because, from (4.6),

$$\begin{aligned}c\|\bar{\mathbf{v}}\|_1^2 = (\nabla\bar{\mathbf{v}},\nabla\bar{\mathbf{v}}) &= -(\bar{\mathbf{w}},\bar{\mathbf{v}}) + (R'(\bar{\mathbf{u}})\bar{\mathbf{v}},\bar{\mathbf{v}})\\ &\leqslant \|\bar{\mathbf{w}}\|\|\bar{\mathbf{v}}\| + \|R'(\bar{\mathbf{u}})\|\|\bar{\mathbf{v}}\|_{L_4}^2\\ &\leqslant \|\bar{\mathbf{w}}\|\|\bar{\mathbf{v}}\| + C(1+\|\bar{\mathbf{u}}\|_1^\delta)\|\bar{\mathbf{v}}\|\|\bar{\mathbf{v}}\|_1,\end{aligned}$$

where we used the Ladyzhenskaya inequality $\|\bar{\mathbf{v}}\|_{L_4}^2 \leqslant C\|\bar{\mathbf{v}}\|\|\bar{\mathbf{v}}\|_1$.

Applying the latest inequalities (4.11), (4.12), (4.13) and (4.14) in (4.10) and using the Schwartz inequality, we get

$$\frac{\mathrm{d}}{\mathrm{d}p}(1+\|\mathbf{u}_h\|+\|\mathbf{v}_h\|+\|\mathbf{w}_h\|) \leqslant C(1+\|\mathbf{u}_h\|+\|\mathbf{v}_h\|+\|\mathbf{w}_h\|)$$

for some constant C depending on $\|\bar{\mathbf{u}}\|_1$, $\|\bar{\mathbf{v}}\|$ and $\|\bar{\mathbf{w}}\|$, noticing that $\|\mathbf{v}_h(t+p)\|_1$ and $\|\mathbf{w}_h(t+p)\|_1$ can be cancelled by the appropriate terms in the left-hand side of (4.10). By the Gronwall lemma, for any sufficiently small p,

$$1+\|\mathbf{u}_h(t+p)\|+\|\mathbf{v}_h(t+p)\|+\|\mathbf{w}_h(t+p)\| \leqslant \mathrm{e}^{Cp}(1+\|\mathbf{u}_h(t)\|+\|\mathbf{v}_h(t)\|+\|\mathbf{w}_h(t)\|).$$

Using the definition of $\mathbf{u}_h$, $\mathbf{v}_h$ and $\mathbf{w}_h$, one realize that a special case of the last inequality is

$$\left\|\frac{\partial^2\mathbf{u}_h(p,t,\bar{\mathbf{u}})}{\partial p^2}\right\| \leqslant \mathrm{e}^{Cp}(1+\|\bar{\mathbf{u}}\|+\|\bar{\mathbf{v}}\|+\|\bar{\mathbf{w}}\|). \qquad \square \tag{4.15}$$

Theorem 4.3. *For any initial value* $\bar{\mathbf{u}} \in V_h$, *if*

$$\bar{\mathbf{v}} = \varDelta_h \bar{\mathbf{u}} + P_h R(\bar{\mathbf{u}}),$$

$$\bar{\mathbf{w}} = \varDelta_h \bar{\mathbf{v}} + P_h (R'(\bar{\mathbf{u}})\bar{\mathbf{v}}),$$

$$\bar{\mathbf{z}} = \varDelta_h \bar{\mathbf{w}} + P_h (R'(\bar{\mathbf{u}})\bar{\mathbf{w}} + R''(\bar{\mathbf{u}})\bar{\mathbf{v}}^2),$$

and there is a constant $\bar{C}$ *such that* $\|\bar{\mathbf{u}}\|_1 \leqslant \bar{C}$, $\|\bar{\mathbf{v}}\| \leqslant \bar{C}$, $\|\bar{\mathbf{w}}\| \leqslant \bar{C}$, $\|\bar{\mathbf{z}}\| \leqslant \bar{C}$, *then the corresponding semi-discrete solution* $\mathbf{u}_h(p, t, \bar{\mathbf{u}})$ *satisfies*

$$\left\| \frac{\partial^3}{\partial p^3} \mathbf{u}_h(p, t, \mathbf{v}) \right\| \leqslant C_M + C_0 \|\bar{\mathbf{u}}\| + C_1 \|\bar{\mathbf{v}}\| + C_2 \|\bar{\mathbf{w}}\| + C_3 \|\bar{\mathbf{z}}\| \tag{4.16}$$

for sufficiently small p and some constants C_M, C_0, C_1, C_2 *and* C_3.

Proof. The proof is similar to the one for the last theorem. □

5. Stability–smoothing indicator and moving attractor

In the proof of the error analysis theorem in the next section, we will split the error between a solution of the weak formulation (4.1) and a numerical solution into five components. In the estimation of each of them, it is crucial to monitor the stability and smoothing behavior of the numerical scheme. To this end, we propose a stability–smoothing indicator, which is computed from the numerical solution.

Definition 5.1. For each node t_i of the time stepping, $t_i = t_0 + \mathrm{i}\tau$, and the value of the numerical solution at t_i, $\bar{\mathbf{u}} = \mathbf{u}_N(t_i)$, let

$$\bar{\mathbf{v}} = \varDelta_h \bar{\mathbf{u}} + P_h R(\bar{\mathbf{u}}),$$

$$\bar{\mathbf{w}} = \varDelta_h \bar{\mathbf{v}} + P_h (R'(\bar{\mathbf{u}})\bar{\mathbf{v}}),$$

$$\bar{\mathbf{z}} = \varDelta_h \bar{\mathbf{w}} + P_h (R'(\bar{\mathbf{u}})\bar{\mathbf{w}} + R''(\bar{\mathbf{u}})\bar{\mathbf{v}}^2).$$

Depending on the necessity, let

$$S_i^2 = (\|\bar{\mathbf{u}}\|_1, \|\bar{\mathbf{v}}\|, \|\bar{\mathbf{w}}\|, \|\varDelta_h \bar{\mathbf{u}}\|)$$

or

$$S_i^3 = (\|\bar{\mathbf{u}}\|_1, \|\bar{\mathbf{v}}\|, \|\bar{\mathbf{w}}\|, \|\bar{\mathbf{z}}\|, \|\varDelta_h \bar{\mathbf{u}}\|).$$

We call the sequence $\{S_i^q \,|\, i \geqslant 0\}$ the stability–smoothing indicator.

If there is a constant C_{ss}, such that each component of S_i^q remains bounded by C_{ss} for all $i > 0$ during the process of the numerical solution, we say that the numerical solution is stable and smoothing.

It will be seen that the stability–smoothing indicator plays a key role in the estimation of both the space approximation error and the time discretization error. In fact, $\|\varDelta_h \bar{\mathbf{u}}\|$ is used to monitor the smoothness of the numerical solution as a function of $x \in \Omega$. All the other components of the stability–smoothing indicator will be needed in monitoring the smoothness of the numerical solution as a function of time t. In practice, when the step size τ is sufficiently small, the $q+1$st component of S_i^q gives us the norm of the qth order time derivative of the semi-discrete solution with initial value $\mathbf{u}_N(t_i)$.

It is easy to find that $\bar{\mathbf{v}} = \partial \mathbf{u}_h(p,t,\bar{\mathbf{u}})/\partial p$, $\bar{\mathbf{w}} = \partial^2 \mathbf{u}_h(p,t,\bar{\mathbf{u}})/\partial p^2$, and $\bar{\mathbf{z}} = \partial^2 \mathbf{u}_h(p,t,\bar{\mathbf{u}})/\partial p^2$. Since we do expect the derivative of the real solution $\partial^q \mathbf{u}(p,t_0,\mathbf{u}_0)/\partial p^q$ to be bounded up to certain q, it is reasonable to expect the components of the stability–smoothing indicator to be bounded. The reason for them to be bounded is smoothing.

Another important concept for error estimation is the moving attractor. For nonlinear evolution equations, it is usually impossible to uniformly control the error between the real solution of a differential equation and a numerical solution, especially for long time intervals. Therefore, it is necessary to consider the error between the numerical solution and an attractive set, in order to obtain a uniform error bound. In fact, because of the undeterministic nature of many strongly nonlinear problems [1], a properly chosen attractive set is very often a better description of the deterministic behavior of the system under consideration. The concepts of attractor, dichotomy and exponential attractor in dynamical system theory are developed to this end. For the error estimation of a numerical solution, it is helpful to generalize the concept of exponential attractors to retain the exponential contraction property but allow the attractive set to change with time. There are also many other benefits from this generalization. The following definitions are similar to those in [11] for ODEs, but modified to work with parabolic problems.

Definition 5.2. Let $\mathscr{M}$ be a one-parameter family of sets in $L^2(\Omega)$,

$$\mathscr{M} = \{M_t \subset L^2(\Omega) | t \geqslant t_0\}.$$

We say that $\mathscr{M}$ is positively invariant under the dynamical system, if for any $\mathbf{v} \in M_t$ and $p>0$, $\mathbf{u}(p,t,\mathbf{v}) \in M_{t+p}$.

Remark. Obviously, if each M_t consists of a single function in $L^2(\Omega)$, then $\mathscr{M}$ can be identified as a solution of Eq. (2.1).

In the next definition and thereafter, we will use the distance from a L_2 function u to a set of functions $M \subset L_2$,

$$d(u,M) = \inf_{w \in M} \|u - w\|.$$

It is easy to verify that

$$d(u,M) \leqslant \|u - v\| + d(v,M)$$

for any function $v \in L_2$.

Definition 5.3. A positively invariant one-parameter family of sets $\mathscr{M}$ in $L^2(\Omega)$ is called a moving attractor, if there exists a real number $s>0$, a real number $\theta_s \in (0,1)$ depending on s, and a one-parameter family of open sets $\mathscr{U} = \{U_t \subset L^2(\Omega) | t \geqslant t_0\}$, positively invariant under the dynamical system, with $M_t \subset U_t$ for all $t \geqslant t_0$, such that for any $\mathbf{v} \in U_t$,

$$d(\mathbf{u}(s,t,\mathbf{v}), M_{t+s}) \leqslant \theta_s d(\mathbf{v}, M_t).$$

$\mathscr{U}$ is called a basin of the moving attractor $\mathscr{M}$.

Obviously, the concept of a moving attractor covers the classical concept of an exponential attractor of a semi-linear parabolic equation. The following example show that the moving attractor concept can be used in other ways for numerical error analysis.

Example 5.4. [11] For a continuous or semi-discrete heat equation with natural boundary condition

$$u_t = \Delta u + f(t), \quad x \in \Omega,$$

$$\nabla u(t,x) \cdot \underline{n} = 0, \quad x \in \partial\Omega$$

and initial condition

$$u(0,x) = g(x),$$

one can consider the moving attractor defined by

$$M_t = \{u(t) + C | C \in \mathbf{R}\}.$$

Here $u(t)$ is the solution. The first eigenvalue λ_0 of the operator Δ is zero, but the second eigenvalue λ_1 is negative. It is easy to show that

$$d(u(t+p), M_{t+p}) \leqslant \mathrm{e}^{\lambda_1 p} d(u(t), M_t).$$

With the help of this moving attractor, one can show that the global error of a numerical solution is uniformly bounded in time if the numerical scheme is convergent and mass-conservative. By mass-conservation we mean that $\int_\Omega u(t,x)\,\mathrm{d}x$ is computed accurately.

6. The error estimation theorem

Now we are ready to state and prove the main error estimation theorem.

Theorem 6.1. *Assume that*

(a) $\mathbf{u}_N(t)$ *is a numerical solution of equation* (4.1), *computed with a finite element method described in Section* 3 *and the discretization in time is consistent to the differential equation with a local error of order* $q = 2$ *or* 3.

(b) *There is a moving attractor* $\mathscr{M}$ *for Eq.* (4.1):

$$d(\mathbf{u}(s,t,\mathbf{v}), M_{t+s}) \leqslant \theta_s d(\mathbf{v}, M_t)$$

for all $t \geqslant t_0$ *and* $\mathbf{v}$ *in the basin.*

(c) *A one-sided Lipschitz condition is satisfied in* $H^1(\Omega)$:

$$(\Delta\mathbf{u} + R(\mathbf{u}) - \Delta\mathbf{v} - R(\mathbf{v}), \mathbf{u} - \mathbf{v}) \leqslant m\|\mathbf{u} - \mathbf{v}\|^2 \tag{6.1}$$

for a real number m.

(d) *The time step size* τ *is chosen so that s is a multiple of* $\tau : s = k\tau$ *for a positive integer k.*

(e) *The stability–smoothing indicator remains bounded.*

Then we have the following global error estimate: For any node of the form $t_0 + ns$ *from* t_0 *to* ∞,

$$d(M_{t_0+ns}, \mathbf{u}_N(ns, t_0, \mathbf{u}_N(t_0))) \leqslant C\frac{s\mathrm{e}^{m^+s}\tau^{q-1}S_M^q + \mathrm{e}^{m^+s}h^2S_H^2}{1-\theta_s} + \theta_s^n d(M_{t_0}, \mathbf{u}_N(t_0)),$$

where $m^+ = \max\{0, m\}$, *and*

$$S_M^q = C_M + \sum_{j=0}^{q} C_j \max_i S_{ij}^q,$$

$$S_H^2 = \max_i \|\varDelta_h \mathbf{u}_N(t_i)\|.$$

Here S_{ij}^q *denotes the jth component of* S_i^q, *while* C_M, C_0, C_1, C_2 *and* C_3 *are the constants given in* (4.4) *or* (4.16), *and* $q = 2$ *or* 3 *is the order of the local truncation error of the time discretization.*

Proof. For any node $t \geqslant t_0$ and the value of the numerical solution $\mathbf{u}_N(t)$ at t, we consider a function $\mathbf{w} \in H_0^1(\Omega) \cap H^2(\Omega)$ given by

$$(\nabla\mathbf{w}, \nabla\mathbf{v}) = -(\varDelta_h\mathbf{u}_N(t), \mathbf{v}) \quad \forall \mathbf{v} \in H_0^1(\Omega). \tag{6.2}$$

From the regularity theory for elliptic equations, we know that

$$\|\mathbf{w}\|_2 \leqslant \|\varDelta_h \mathbf{u}_N(t)\| \leqslant S_H^2.$$

Note that, if $\mathbf{v}$ is restricted in V_h in (6.2), we can obtain

$$(\nabla\mathbf{u}_N(t), \nabla\mathbf{v}) = -(\Delta\mathbf{w}, \mathbf{v}) \quad \forall \mathbf{v} \in V_h \tag{6.3}$$

by integration by parts. Since

$$(\nabla\mathbf{w}, \nabla\mathbf{v}) = -(\Delta\mathbf{w}, \mathbf{v}) \quad \forall \mathbf{v} \in H_0^1(\Omega), \tag{6.4}$$

we realize that $u_N(t)$ is the Galerkin finite element approximation of the solution $\mathbf{w}$ of Eq. (6.4). The standard finite element error analysis and the duality argument then tell us that

$$\|\mathbf{w} - \mathbf{u}_N(t)\| \leqslant Ch^2\|\mathbf{w}\|_2. \tag{6.5}$$

Now, we split the error between the numerical solution at time $t+s$ and the moving attractor into five parts,

$$d(M_{t+s}, \mathbf{u}_N(s, t, \mathbf{u}_N(t)))$$
$$\leqslant d(M_{t+s}, \mathbf{u}(s, t, \mathbf{u}_N(t))) \tag{6.6}$$
$$+ \|\mathbf{u}(s, t, \mathbf{u}_N(t)) - \mathbf{u}(s, t, \mathbf{w})\| \tag{6.7}$$
$$+ \|\mathbf{u}(s, t, \mathbf{w}) - \mathbf{u}_h(s, t, \mathbf{w})\| \tag{6.8}$$
$$+ \|\mathbf{u}_h(s, t, \mathbf{w}) - \mathbf{u}_h(s, t, \mathbf{u}_N(t))\| \tag{6.9}$$
$$+ \|\mathbf{u}_h(s, t, \mathbf{u}_N(t)) - \mathbf{u}_N(s, t, \mathbf{u}_N(t))\|. \tag{6.10}$$

Since $\mathscr{M}$ is a moving attractor, the distance in (6.6) can be estimated by

$$d(M_{t+s}, \mathbf{u}(s,t,\mathbf{u}_N(t))) \leqslant \theta_s d(M_t, \mathbf{u}_N(t)). \tag{6.11}$$

Due to the one-sided Lipschitz condition and (6.5), the difference in (6.7) satisfies

$$\|\mathbf{u}(s,t,\mathbf{u}_N(t)) - \mathbf{u}(s,t,\mathbf{w})\| \leqslant \mathrm{e}^{ms}\|\mathbf{u}_N(t) - \mathbf{w}\| \leqslant C\mathrm{e}^{ms}h^2\|\mathbf{w}\|_2. \tag{6.12}$$

For the difference in (6.8), we observe that it is the error between an exact solution and a semi-discrete solution, both having the H^2-smooth initial value $\mathbf{w}$. In [8], it is proven that this error is bounded by

$$\|\mathbf{u}(s,t,\mathbf{w}) - \mathbf{u}_h(s,t,\mathbf{w})\| \leqslant Ch^2\|\mathbf{w}\|_2. \tag{6.13}$$

Observing that the one-sided Lipschitz condition is also satisfied by the semi-discrete problem (4.2), similar to (6.12), we have, for the difference in (6.9),

$$\|\mathbf{u}_h(s,t,\mathbf{w}) - \mathbf{u}_h(s,t,\mathbf{u}_N(t))\| \leqslant \mathrm{e}^{ms}\|\mathbf{w} - \mathbf{u}_N(t)\| \leqslant C\mathrm{e}^{ms}h^2\|\mathbf{w}\|_2. \tag{6.14}$$

In fact, the one-sided Lipschitz condition (3.2) is valid in V_h, since V_h is a subspace of $H_0^1(\Omega)$.

As for the difference in (6.10), it is the error between the ODE resulting from (4.2) and the numerical solution. To estimate the error in approximating this ODE from time t to $t+s$, we need to use the technique developed in [11]. Since the local error of the time discretization is of order $q=2$ or 3, for each $t_i \in [t, t+s)$, we have

$$\|\mathbf{u}_N(\tau, t_i, \mathbf{u}_N(t_i)) - \mathbf{u}_h(\tau, t_i, \mathbf{u}_N(t_i))\| \leqslant C\tau^q \max_{p\in[0,\tau]} \left\| \frac{\partial^q}{\partial p^q}\mathbf{u}_h(p, t_i, \mathbf{u}_N(t_i)) \right\|. \tag{6.15}$$

Based on the stability–smoothing indicator and the smoothing property (4.4) or (4.16), we know that

$$\max_{p\in[0,\tau]} \left\| \frac{\partial^q}{\partial p^q}\mathbf{u}_h(p, t_i, \mathbf{u}(t_i)) \right\| \leqslant S_M^q.$$

Therefore,

$$\begin{aligned}
&\|\mathbf{u}_h(\tau, t_i, \mathbf{u}_h(t_i)) - \mathbf{u}_N(\tau, t_i, \mathbf{u}_N(t_i))\| \\
&\quad \leqslant \|\mathbf{u}_h(\tau, t_i, \mathbf{u}_h(t_i)) - \mathbf{u}_h(\tau, t_i, \mathbf{u}_N(t_i))\| + \|\mathbf{u}_h(\tau, t_i, \mathbf{u}_N(t_i)) - \mathbf{u}_N(\tau, t_i, \mathbf{u}_N(t_i))\| \\
&\quad \leqslant \mathrm{e}^{m\tau}\|\mathbf{u}_h(t_i) - \mathbf{u}_N(t_i)\| + C\tau^q S_M^q.
\end{aligned} \tag{6.16}$$

Recall that $s = k\tau$ and identify each node $t_i \in [t, t+s]$ with $t + j\tau$ for some $j \geqslant 0$. By using (6.16) repeatedly, we obtain

$$\begin{aligned}
&\|\mathbf{u}_h(s,t,\mathbf{u}_N(t)) - \mathbf{u}_N(s,t,\mathbf{u}_N(t))\| \\
&\quad = \|\mathbf{u}_h(k\tau,t,\mathbf{u}_N(t)) - \mathbf{u}_N(k\tau,t,\mathbf{u}_N(t))\| \\
&\quad \leqslant \|\mathbf{u}_h(\tau, t+k\tau-\tau, \mathbf{u}_h(k\tau-\tau, t, \mathbf{u}_N(t))) - \mathbf{u}_h(\tau, t+k\tau-\tau, \mathbf{u}_N(t+k\tau-\tau))\| \\
&\qquad + \|\mathbf{u}_h(\tau, t+k\tau-\tau, \mathbf{u}_N(t+k\tau-\tau)) - \mathbf{u}_N(\tau, t+k\tau-\tau, \mathbf{u}_N(t+k\tau-\tau))\| \\
&\quad \leqslant \mathrm{e}^{m\tau}\|\mathbf{u}_h((k-1)\tau, t, \mathbf{u}_N(t)) - \mathbf{u}_N((k-1)\tau, t, \mathbf{u}_N(t))\| + C\tau^q S_M^q \\
&\quad \leqslant \cdots \leqslant \mathrm{e}^{jm\tau}\|\mathbf{u}_h((k-j)\tau, t, \mathbf{u}_N(t)) - \mathbf{u}_N((k-j)\tau, t, \mathbf{u}_N(t))\| \\
&\qquad + (1 + \mathrm{e}^{m\tau} + \cdots + \mathrm{e}^{(j-1)m\tau})C\tau^q S_M^q \\
&\quad \leqslant \cdots \leqslant (1 + \mathrm{e}^{m\tau} + \cdots + \mathrm{e}^{(k-1)m\tau})C\tau^q S_M^q.
\end{aligned}$$

If $m \leqslant 0$,

$$\tau(1 + e^{m\tau} + \cdots + e^{(k-1)m\tau}) \leqslant k\tau = s.$$

If $m > 0$, by using the simple inequality $1 \leqslant (e^x - 1)/x \leqslant e^x$ for $x > 0$, we know

$$\tau(1 + e^{m\tau} + \cdots + e^{(k-1)m\tau}) = \tau \frac{e^{ms} - 1}{e^{m\tau} - 1} \leqslant \frac{e^{ms} - 1}{m} = s\frac{e^{ms} - 1}{ms} \leqslant s e^{ms}.$$

In either case, we have

$$\|\mathbf{u}_h(s, t, \mathbf{u}_N(t)) - \mathbf{u}_N(s, t, \mathbf{u}_N(t))\| \leqslant C\tau^{q-1} s e^{m^+ s} S_M^q. \tag{6.17}$$

Combining the five term splitting (6.6) to (6.10) with (6.11), (6.12), (6.13), (6.14) and (6.17), we get

$$d(M_{t+s}, \mathbf{u}_N(s, t, \mathbf{u}_N(t))) \leqslant \theta_s d(M_t, \mathbf{u}_N(t)) + C(s e^{m^+ s} \tau^{q-1} S_M^q + e^{m^+ s} h^2 S_H^2).$$

Now, by repeatedly using the last inequality,

$$\begin{aligned}
&d(M_{t_0+ns}, \mathbf{u}_N(ns, t_0, \mathbf{u}_N(t_0))) \\
&\quad \leqslant d(M_{t_0+(n-1)s+s}, \mathbf{u}(s, t_0 + (n-1)s, \mathbf{u}_N((n-1)s, t_0, \mathbf{u}_N(t_0)))) \\
&\qquad + \|\mathbf{u}(s, t_0 + (n-1)s, \mathbf{u}_N((n-1)s, t_0, \mathbf{u}_N(t_0))) \\
&\qquad - \mathbf{u}_N(s, t_0 + (n-1)s, \mathbf{u}_N((n-1)s, t_0, \mathbf{u}_N(t_0)))\| \\
&\quad \leqslant \theta_s d(M_{t_0+(n-1)s}, \mathbf{u}_N((n-1)s, t_0, \mathbf{u}_N(t_0))) + C(s e^{m^+ s} \tau^{q-1} S_M^q + e^{m^+ s} h^2 S_H^2) \\
&\quad \leqslant \cdots \leqslant C(1 + \theta_s + \cdots + \theta_s^{n-1})(s e^{m^+ s} \tau^{q-1} S_M^q + e^{m^+ s} h^2 S_H^2) + \theta_s^n d(M_{t_0}, \mathbf{u}_N(t_0)) \\
&\quad \leqslant C \frac{s e^{m^+ s} \tau^{q-1} S_M^q + e^{m^+ s} h^2 S_H^2}{1 - \theta_s} + \theta_s^n d(M_{t_0}, \mathbf{u}_N(t_0)). \qquad \square
\end{aligned}$$

7. Numerical experiments

Example 7.1. The Chaffee–Infante equation

$$\dot{\mathbf{u}} = \Delta \mathbf{u} + 15(\mathbf{u} - \mathbf{u}^3) \tag{7.1}$$

with

$$\mathbf{u} = 0 \quad x \in \partial\Omega$$

is solved in the domain Ω as shown in Fig. 1. The coordinates of the corners of the domain are $(0, 0)$, $(1, 0)$, $(1.2, 1.5)$, $(0, 1)$, $(0.2, 0.3)$, $(0.8, 0.2)$ and $(0.2, 0.8)$. Piecewise linear elements are used and the time stepping is implicit on the diffusion term but explicit on the reaction term:

$$\left(\frac{\mathbf{u}_N^{(k+1)} - \mathbf{u}_N^{(k)}}{\tau}, \mathbf{v} \right) + (\nabla \mathbf{u}_N^{(k+1)}, \nabla \mathbf{v}) = (R(\mathbf{u}_N^{(k)}), \mathbf{v}) \tag{7.2}$$

for all $\mathbf{v} \in V_h$, where $R(u) = 15(u - u^3)$. From [1], we know that there is a unique nontrivial, positive, stable, steady-state solution for (7.1) and an exponential attractor containing this steady-state solution.

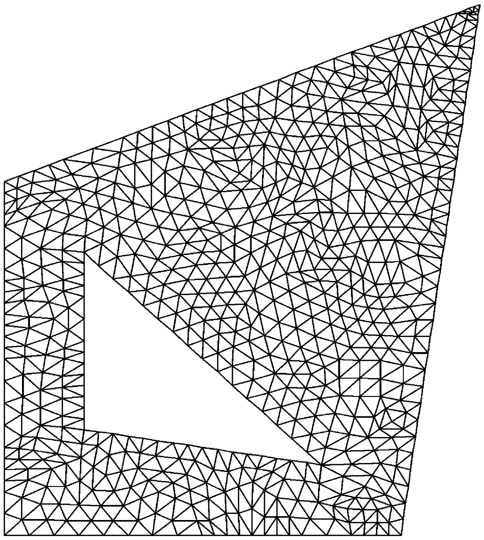

Fig. 1. Domain and triangulation of example 7.1.

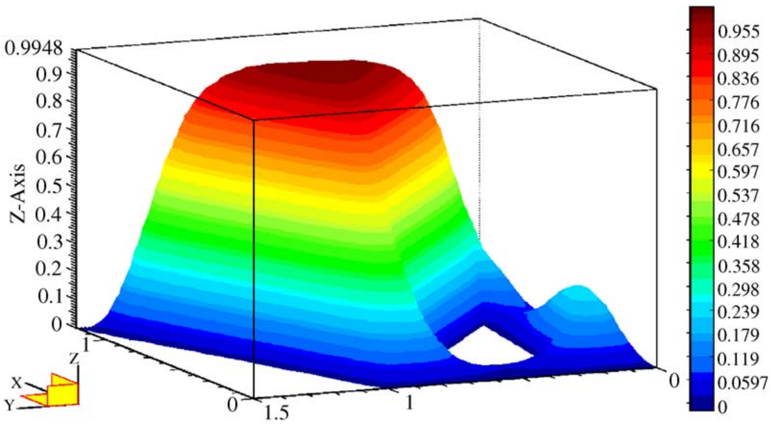

Fig. 2. The stable static solution of (7.1) in Ω.

With any positive initial value, a time-dependent solution will asymptotically converge to the steady-state solution. According to our computation, the contour map of this steady-state solution is given in Fig. 2.

The stability–smoothing indicator is computed during the process of the numerical solution. Fig. 3 shows the second component $(\partial \mathbf{u}_h/\partial t)$ and the third component $(\partial^2 \mathbf{u}_h/\partial t^2)$ of the stability–smoothing indicator as functions of time. Throughout the computation, the stability–smoothing indicator was bounded. Moreover, the second and third components asymptotically converged to zero. This is exactly the smoothing effect of the numerical scheme.

Example 7.2. For the Chaffee–Infante equation above, it is not very easy to obtain the contraction rate of an exponential attractor. Besides, the one-sided Lipschitz constant is $m = 15$ if one considers the worst case where $\mathbf{u}$ and $\mathbf{v}$ in (6.1) is allowed to be arbitrarily close to zero. To avoid these troubles and show

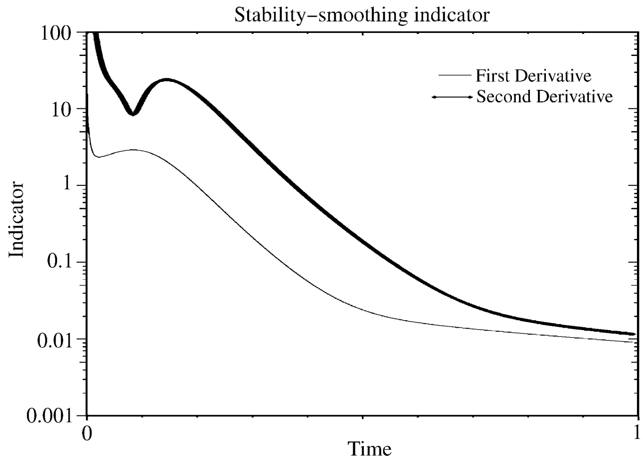

Fig. 3. The stability–smoothing indicator for Example 1.

an example in which we can obtain a posteriori error estimate, we consider the following equation with a monotone operator on the right-hand side:

$$\dot{\mathbf{u}} = \Delta \mathbf{u} + 15(f(x, y, t) - \mathbf{u}^3) \tag{7.3}$$

with a continuous function $f(x, y, t)$, the Dirichlet boundary condition

$$\mathbf{u} = 0 \quad x \in \partial\Omega$$

and the same domain Ω as shown in Example 1.

Again, piecewise linear elements are used and the time stepping is implicit on the diffusion term but explicit on the reaction term:

$$\left(\frac{\mathbf{u}_N^{(k+1)} - \mathbf{u}_N^{(k)}}{\tau}, \mathbf{v}\right) + (\nabla \mathbf{u}_N^{(k+1)}, \nabla \mathbf{v}) = (R(x, y, t, \mathbf{u}_N^{(k)}), \mathbf{v}) \tag{7.4}$$

for all $\mathbf{v} \in V_h$, where $R(x, y, t, u) = 15(f(x, y, t) - u^3)$. Obviously, the one-sided Lipschitz constant is $m = -\lambda_1$, where $\lambda_1 > 0$ is the smallest eigenvalue of $-\Delta$ on the domain Ω with the homogeneous Dirichlet boundary condition. Since λ_1 is always positive under the Dirichlet boundary condition, Theorem 3.3 tells us that every solution $\mathbf{u}(t)$ of (7.3) is a moving attractor, that is, $M_t = \{\mathbf{u}(t)\}$ in Definition 5.3.

Fig. 4 shows the second component $(\partial \mathbf{u}_h/\partial t)$ and the third component $(\partial^2 \mathbf{u}_h/\partial t^2)$ of the stability–smoothing indicator for our numerical solution, in which $f(x, y, t) = 0.2 + \sin(5t)$. Throughout the computation, the stability–smoothing indicator was bounded. In this example, they are not expected to decay to zero, due to the existence of the "controlling term" $f(x, y, t)$.

It is easy to see that the second time derivative is bounded by 80 for most of the time. We also computed that the norm of the discrete Laplacian of the numerical solution, $\|\Delta \mathbf{u}_N\|$, is bounded by 50 during the computation. Since the error is carefully controlled near the initial time, Theorem 6.1 guarantees that the

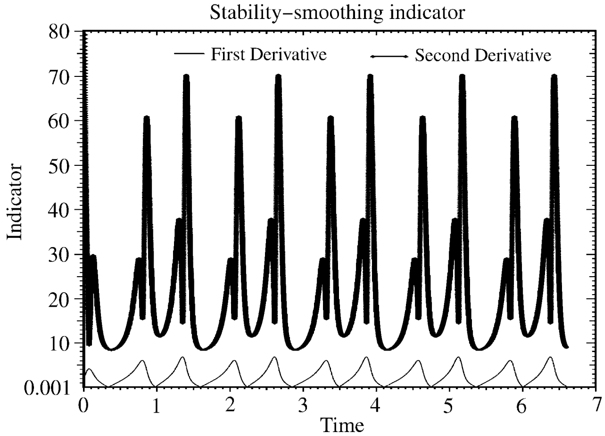

Fig. 4. The stability–smoothing indicator for Example 2

error between the numerical solution and the real solution is bounded by

$$\|\mathbf{u}(ns, t_0, \mathbf{u}_0) - \mathbf{u}_N(ns, t_0, \mathbf{u}_{N0})\| \leqslant C\frac{80s\tau + 50h^2}{1 - e^{-\lambda_1 s}} + e^{-\lambda_1 ns}\|\mathbf{u}_0 - \mathbf{u}_{N0}\|. \tag{7.5}$$

One can actually compute the constant C here very easily just by following what has contributed to the constant in the proof of Theorem 6.1. In fact, if we use C_{sp} to denote the constant in (6.5) and use C_{ti} for the constant in (6.17), then the constant C here is bounded by $3C_{sp} + C_{ti}$. The three copies of C_{sp} correspond to the second, third and fourth term in the five term error splitting, while the one C_{ti} is from the last term. Here we only want to point out that it is possible to obtain a posteriori error estimates. If one tries harder in practice, one can certainly get a sharper bound for this constant.

References

[1] A. Eden, C. Foias, B. Nicolaenko, R. Temam, Exponential Attractors for Dissipative Evolution Equations, Wiley, New York, 1994.
[2] K. Eriksson, C. Johnson, Adaptive finite element methods for parabolic problems. V. Long-time integration, SIAM J. Numer. Anal. 32 (1995) 1750–1763.
[3] K. Eriksson, C. Johnson, S. Larsson, Adaptive finite element methods for parabolic problems. VI. Analytic semigroups, SIAM J. Numer. Anal. 35 (1998) 1315–1325.
[4] D. Estep, M. Larson, R. Williams, Estimating the error of numerical solutions of systems of nonlinear reaction–diffusion equations, Mem. Amer. Math. Soc. 696 (2000) 1–109.
[5] E. Hairer, S. Nørsett, G. Wannar, Solving Ordinary Differential Equations I: Nonstiff Problems, Springer, New York, 1987.
[6] E. Hairer, G. Wannar, Solving Ordinary Differential Equations II: Stiff Problems and Differential-Algebraic Equations, Springer, New York, 1991.
[7] J. Heywood, R. Rannacher, Finite element approximation of the Navier–Stokes problem, Part II. Stability of solutions and error estimates uniform in time, SIAM J. Numer. Anal. 23 (1986) 750–777.
[8] S. Larsson, Numerical analysis of semilinear parabolic problems, in: M. Ainsworth, J. Levesley, M. Marletta (Eds.), The Graduate Student's Guide to Numerical Analysis '98, SSCM, vol. 26, Springer, Berlin, 1999, pp. 83–117.
[9] J. Smoller, Shock Waves and Reaction–Diffusion Equations, Springer, New York, 1983.
[10] A. Stuart, A. Humphries, Dynamical Systems and Numerical Analysis, Cambridge University Press, Cambridge, 1996.
[11] T. Sun, R. Ewing, Long-time error estimation and a stability indicator, Dyn. Contin. Discrete Impuls. Systems Ser. B Appl. Algorithms 9 (2002) 115–129.